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Augmentations of consistent partial orders for the one-machine total tardiness problem*

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Abstract

A partial order on a job set is called consistent, if it has a linear extension which is an optimal solution to the total tardiness problem of the job set. The concept of proper augmentations of consistent partial orders is based on Emmons' well-known dominance theorem. In this paper, we address the question of whether the proper augmentation of a consistent partial order always results in a partial order which is also consistent. By giving an example, we show that this need not be true in general. However, as the main result of this paper, we prove that the answer to this question is affirmative for the normal procedure, i.e., the procedure of proper augmentations beginning from “null”. Therefore, this paper closes the gap between Emmons' dominance theorem and the normal procedure of augmentations of partial orders.

Keywords: One-machine scheduling; Total tardiness problem; Consistent partial order; Proper augmentation; Dominance condition, Dominance theorem

1. Introduction

Given n jobs. For job i ($i = 1, 2, \dots, n$), let p_i be its processing time ($p_i > 0$) and d_i ($-\infty < d_i < \infty$) be its due date. In the case of one-machine processing, when a job sequence $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of the set of jobs $N = \{1, 2, \dots, n\}$ is prescribed, then the completion time $C_i(\sigma)$ of job $i \in N$ and its tardiness $T_i(\sigma) = \max(0, C_i(\sigma) - d_i)$ are determined. The total tardiness problem of the job set N is as follows:

$$TTP(N): \min_{\sigma} T(\sigma) = \sum_{i \in N} \max(0, C_i(\sigma) - d_i). \quad (1.1)$$

A partial order on set N is a binary relation on N , i.e., a subset Q of $N^2 = \{(i, j) | i, j \in N\}$, which is reflexive, transitive and antisymmetric. Thus,

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$(i, j) \in Q (i \neq j)$ is used to denote “job i precedes job j ” for problem (1.1). Besides, $(i, i) \in Q$ holds always from the reflexivity of Q . In this paper we prefer the notation $(i, j) \in Q$ to another often used notation $i \leq j$ since some augmentations of Q are studied.

Let PO stand for the terminology “partial order”, and let $PO(N)$ stand for the collection of all partial orders on N . Assume that $Q \in PO(N)$ and that $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ is a sequence of N . If $\sigma^{-1}(i) \leq \sigma^{-1}(j)$ holds for any $(i, j) \in Q$, then σ is called a linear extension of Q ; in other words, σ is said to satisfy Q . Also, let $LE(Q)$ denote the collection of all linear extensions of Q .

Definition 1. Given $Q \in PO(N)$. For any job $i \in N$, define

$$C_i^-(Q) = \sum_{(s,i) \in Q} p_s, \quad C_i^+(Q) = \sum_{(i,s) \notin Q} p_s + p_i. \quad (1.2)$$

$C_i^-(Q)$ and $C_i^+(Q)$ are called the earliest and the latest completion time of job i based on Q , because it is easy to prove that $C_i^-(Q)$ and $C_i^+(Q)$ are minimum and maximum of $C_i(\sigma)$ for $\sigma \in LE(Q)$, respectively. As a part of this conclusion, it holds that

$$\forall \sigma \in LE(Q), \quad C_i^-(Q) \leq C_i(\sigma) \leq C_i^+(Q). \quad (1.3)$$

Definition 2. Given $Q \in PO(N)$. Define three subsets of N^2 as follows:

$$IC(Q) = \{(i, j) | i, j \in N, i \neq j, p_i \leq p_j, d_i \leq \max(d_j, C_j^-(Q))\}, \quad (1.4)$$

$$BS(Q) = \{(i, j) | i, j \in N, i \neq j, d_j \geq \min(C_i^+(Q), \max(d_i, C_i^+(Q) - p_j))\}, \quad (1.5)$$

$$DC(Q) = IC(Q) \cup BS(Q). \quad (1.6)$$

The conditions for a pair (i, j) in $IC(Q)$, in $BS(Q)$ and in $DC(Q)$ are called the interchange condition, the backward-shift condition and the dominance condition based on Q , respectively.

To make the purpose of this paper clear, we restate the main results of Emmons [2] in our notations. Conclusions and proofs of the following two lemmas can be found in Emmons [2]. Also some simplified proofs of them are given in the Appendix of this paper.

Interchange Lemma (Emmons [2]). Assume $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, $\sigma^{-1}(j) < \sigma^{-1}(i)$, $p_i \leq p_j$ and $d_i \leq \max(d_j, C_j(\sigma))$. Let α be the job sequence obtained from σ by interchanging job i and job j . Then $T(\alpha) \leq T(\sigma)$.

Backward-Shift Lemma (Emmons [2]). Assume $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, $\sigma^{-1}(j) < \sigma^{-1}(i)$ and $d_j \geq \min(C_i(\sigma), \max(d_i, C_i(\sigma) - p_j))$. Let β be the job sequence obtained from σ by shifting job j backward to be immediately after job i . Then $T(\beta) \leq T(\sigma)$.

Definition 3. A transformation on job sequence σ in accordance with either one of the above two lemmas is called Emmons' transformation on σ for job pair (i, j) .

Due to (1.3) and the above two lemmas, we have the following.

Transformation Theorem (Emmons [2]). *Given $Q \in PO(N)$, $\sigma \in LE(Q)$, $\sigma^{-1}(j) < \sigma^{-1}(i)$ and $(i, j) \in DC(Q)$. Let π be the job sequence obtained from σ by taking the Emmons' transformation on σ for job pair (i, j) . Then $T(\pi) \leq T(\sigma)$.*

Definition 4. Given $Q \in PO(N)$. If there exists $\sigma \in LE(Q)$ such that σ is an optimal solution for $TTP(N)$, then Q is called an optimality-consistent PO of $TTP(N)$, in short, Q is called a consistent PO of $TTP(N)$.

The following theorem can be obtained by using the Transformation Theorem.

Dominance Theorem (Emmons [2]). *If Q is a consistent PO of $TTP(N)$, and $(i, j) \in DC(Q)$, then $TTP(N)$ possesses an optimal solution π with $\pi^{-1}(i) < \pi^{-1}(j)$.*

The Dominance Theorem might suggest PO augmentations by adding the job pair (i, j) to Q successively so as to make the consistent partial orders “stronger” than Q . In this way, the Dominance Theorem “plays a major role in the algorithms” for the total tardiness problem as analyzed in [1]. Therefore, Emmons [2] is “among the most important” papers written for the total tardiness problem as pointed out in Lawler [3].

When the assumptions of the Dominance Theorem are satisfied and (i, j) is added to Q , does this always result in a new consistent PO ? This remains an important question, which becomes clear when an analysis is made in relation to the proof of the Dominance Theorem. Since Q is a consistent PO of $TTP(N)$, there is an optimal solution $\sigma \in LE(Q)$. In case of $\sigma^{-1}(j) < \sigma^{-1}(i)$, due to $(i, j) \in DC(Q)$ and the Transformation Theorem, it holds that $T(\pi) \leq T(\sigma)$, where π is the sequence obtained from σ by taking the Emmons' transformation on σ for (i, j) . Now $\pi^{-1}(i) < \pi^{-1}(j)$ and π is also an optimal solution, thus the conclusion is obtained. But π may disobey some precedence relations in Q which σ satisfies. So it is not immediately clear whether (i, j) can be added to Q to get a new consistent PO .

As described in some papers, it seems that the answer to this question is considered commonly to be affirmative for the procedure of PO augmentations using the dominance condition and beginning from “null”. Nevertheless, it was pointed out in Lin [4] that “we conjecture this way is correct, and we are expecting a rigorous proof”. To the author's understanding, Lin [4] is the first paper which posed the above question clearly. Also to the author's understanding, Emmons [2] mentioned the above question already implicitly. According to the exposition in [2], consistency of a PO is an optimality property which is “existential” but not “universal”. It was said in [2] that “existential properties cannot generally be accumulated”. On the other hand,

it was said too in [2] that “the properties we shall derive can be” accumulated, “by making as many interchanges as necessary to obtain an optimal schedule with all the properties”. In relation to these words, by asking the above question we intend to investigate why “as many interchanges as necessary” can be made, i.e., why all transformations can be made to obtain an optimal schedule with all the properties.

In this paper, we define the proper augmentation in Section 2 as the procedure of adding (i, j) to Q under the assumptions of the Dominance Theorem.

As the main result of this paper, we prove in Section 5 that any partial order obtained by proper augmentations beginning from “null” must be a consistent PO . Also we give an example in Section 2 to show that a partial order obtained by the proper augmentation from a consistent PO can be non-consistent in general.

Section 2 is about the proper augmentations. Some restricted transitivity of the dominance condition is proved in Section 3, and it is applied to get some properties of proper augmentations. The induced partial order on a subset is discussed in relation to proper augmentations in Section 4. Section 2–Section 4 are preparations for proving the main result of this paper, i.e., Theorem 3 in Section 5. Finally, Section 6 contains some concluding remarks.

2. Proper Augmentations of Consistent Partial Orders

Definition 5. Given $Q \in PO(N)$. Assume that i and j are incomparable in Q , i.e., $i, j \in N$ and $(i, j), (j, i) \notin Q$. Define $Q \oplus (i, j) = Q \cup \{(r, s) | (r, i), (j, s) \in Q\}$.

Thus $Q \oplus (i, j)$ is obtained from Q by adding (i, j) and other ordered pairs implied by transitivity. Two simple lemmas are stated without proof.

Lemma 1. $Q \oplus (i, j) \in PO(N)$.

Assume that $Q \in PO(N)$. An element k of N is called maximal w.r.t Q if there does not exist $j \in N$ such that $j \neq k$ and $(k, j) \in Q$. Let $\max(Q)$ denote the subset of N which is composed of all maximal elements w.r.t Q .

Lemma 2. If $k \in \max(Q)$ and $k \neq i$, then $k \in \max(Q \oplus (i, j))$.

Definition 6. Assume that $Q \in PO(N)$, $R = Q \oplus (i, j)$ and $(i, j) \in DC(Q)$. The procedure from Q to R is called a proper augmentation from Q , and R is called a proper augmentation PO of Q , denoted by $R = PAPO(Q)$. Also, (i, j) is called the primitive arc of the proper augmentation from Q to R .

Definition 7. $Q_{00} = Q_{00}(N) = \{(i, i) | i \in N\}$ is called null PO on N , or null.

Example 1. This example shows that when Q is a consistent PO of $TTP(N)$, $R = PAPO(Q)$ is not necessarily a consistent PO of $TTP(N)$.

$N = \{1, 2, 3\}$. $(p_1, d_1) = (20, 10)$, $(p_2, d_2) = (10, 30)$, $(p_3, d_3) = (20, 11)$. Let $\alpha = (1, 2, 3)$ and $\beta = (3, 2, 1)$. The job data are such that $T(\alpha) = T(\beta) = 49$, and $T(\sigma) = 59$ for any $\sigma \neq \alpha, \beta$. Now let $Q = Q_{00} \cup \{(3, 2)\}$. Q is a consistent PO , since $\beta = (3, 2, 1) \in LE(Q)$ and β is an optimal solution of $TTP(N)$. Because $(1, 3) \in IC(Q)$, let $R = Q \oplus (1, 3) = (Q_{00} \oplus (3, 2)) \oplus (1, 3)$. R has only one linear extension $\gamma = (1, 3, 2)$ which is not an optimal solution, so R is not a consistent PO .

Definition 8. Given $Q_0 \in PO(N)$ and $Q^* = \{Q_0, Q_1, \dots, Q_m\}$. If $Q_{s+1} = PAPO(Q_s)$, where $s = 0, 1, \dots, m-1$, then Q^* is called a proper augmentation PO system on N from Q_0 , in short, Q^* is called a $PAPO$ system. Also, m is called the degree of Q^* , and each Q_s in Q^* is called a proper augmentation PO with degree s from Q_0 .

Lemma 3. Assume that $Q^* = \{Q_0, Q_1, \dots, Q_m\}$ is a $PAPO$ system on N and that $j \in \max(Q_0) \setminus \max(Q_m)$. Then there exist a unique $t \in [0, m)$ and a unique arc $(j, k) \in Q_m$ such that $j \in \max(Q_t) \setminus \max(Q_{t+1})$, and (j, k) is the primitive arc of the proper augmentation from Q_t to Q_{t+1} .

Proof. For $s = 0, 1, \dots, m$, $\max(Q_s)$ is non-increasing since Q_s is increasing. Thus $j \in \max(Q_0) \setminus \max(Q_m)$ implies a unique $t \in [0, m)$ such that $j \in \max(Q_t) \setminus \max(Q_{t+1})$. Now for the proper augmentation from Q_t to Q_{t+1} , the primitive arc must be an arc from j , otherwise according to Lemma 2 we would have that $j \in \max(Q_{t+1})$. \square

Definition 9. Assume that $Q^* = \{Q_0, Q_1, \dots, Q_m\}$ is a $PAPO$ system from null, i.e., $Q_0 = Q_{00}(N)$. For any $j \notin \max(Q_m)$, (j, k) in Lemma 3 is called the earliest primitive arc from j w.r.t Q^* . Therefore, there exists a unique path from j to some $r \in \max(Q_m)$, which consists only of earliest primitive arcs. This path is called the primitive path from j w.r.t Q^* , and r is called the primitive terminal of j w.r.t Q^* . Finally, when $j \in \max(Q_m)$, then the primitive path from j is defined as the empty set, and the primitive terminal of j is defined as j itself. In either one of the above two cases, the primitive terminal of j is denoted by $r = \text{terminal}(j)$.

3. Restricted transitivity of the dominance condition

In this section, we prove that under certain restrictions, transitivity holds for the dominance condition defined by (1.4) and (1.5). This will be used in the further discussion on $PAPO$ systems in Section 5. Obviously, the following lemma holds.

Lemma 4. Given $Q \in PO(N)$. If $(j, k) \in Q$, then

$$C_j^-(Q) \leq C_k^-(Q) - p_k, \quad C_j^+(Q) \leq C_k^+(Q) - p_k.$$

Lemma 5. Given $Q, Q' \in PO(N)$ and $Q' \subset Q$. For any $i \in N$, it holds that

$$C_i^-(Q') \leq C_i^-(Q), \quad C_i^+(Q') \geq C_i^+(Q). \quad (3.1)$$

Furthermore, if $(i, k) \notin Q$ and $(j, k) \in Q \setminus Q'$, then

$$C_i^+(Q) \geq C_k^-(Q') + p_i + p_j. \quad (3.2)$$

Proof. Inequality (3.1) is obvious. To prove (3.2), we consider the following two expressions:

$$C_k^-(Q') = \sum_{(s, k) \in Q'} p_s, \quad (3.3)$$

$$C_i^+(Q) = \sum_{(i, s) \notin Q} p_s + p_i. \quad (3.4)$$

The proof of (3.2) is based on the following three observations:

- (i) Because $(i, k) \notin Q$ implies $(i, k) \notin Q'$, p_i is not included in (3.3), but p_i is included in (3.4) explicitly.
- (ii) Due to $(j, k) \notin Q'$, p_j is not included in (3.3), but p_j is included in (3.4), since $(i, j) \notin Q$ is implied by $(j, k) \in Q$ and $(i, k) \notin Q$.
- (iii) Any p_s in (3.3) must be in (3.4) too, since $(i, s) \notin Q$ is implied by $(s, k) \in Q' \subset Q$ and $(i, k) \notin Q$. \square

The following lemma concerns the median value $\text{med}(\cdot, \cdot, \cdot)$ of three real numbers.

Lemma 6. Assume $a \leq c$, then it holds that

$$\text{med}(a, b, c) = \min(\max(a, b), c) = \max(a, \min(b, c)). \quad (3.5)$$

Assume $a \leq c$, $a' \leq c'$, $a \leq a'$ and $c \leq c'$. Also assume that either $b \leq \max(a', b')$ or $\min(b, c) \leq b'$ holds. Then

$$\text{med}(a, b, c) \leq \text{med}(a', b', c'). \quad (3.6)$$

Proof. Eq. (3.5) can be checked easily for cases: (i) $b < a$, (ii) $a \leq b \leq c$, (iii) $b > c$.

Now we prove (3.6) in case of $b \leq \max(a', b')$. Also $a \leq \max(a', b')$ is implied by $a \leq a'$. Thus $\max(a, b) \leq \max(a', b')$. Combining this inequality with $c \leq c'$ and using (3.5), we get (3.6). The proof of (3.6) in case of $\min(b, c) \leq b'$ is similar. \square

Theorem 1. Given $Q', Q \in PO(N)$ and $Q' \subset Q$. Assume $(i, k) \notin Q$, $(j, k) \in Q$ and $j \in \max(Q')$. Then it holds that

$$(i, j) \in DC(Q), (j, k) \in DC(Q') \Rightarrow (i, k) \in DC(Q).$$

Proof. The following four cases are discussed:

Case (i): $(i, j) \in IC(Q)$ and $(j, k) \in IC(Q')$. We have inequalities as follows:

$$p_i \leq p_j, \quad d_i \leq \max\{d_j, C_j^-(Q)\}, \quad p_j \leq p_k, \quad d_j \leq \max\{d_k, C_k^-(Q')\}.$$

Thus it follows that

$$p_i \leq p_k, \quad d_i \leq \max\{d_k, C_k^-(Q'), C_j^-(Q)\}. \quad (3.7)$$

Due to Lemma 5 it holds that $C_k^-(Q') \leq C_k^-(Q)$. And due to Lemma 4, it follows from $(j, k) \in Q$ that $C_j^-(Q) < C_k^-(Q)$. From these results and (3.7) we obtain that $p_i \leq p_k$ and $d_i \leq \max\{d_k, C_k^-(Q)\}$, i.e., $(i, k) \in IC(Q) \subseteq DC(Q)$.

Case (ii): $(i, j) \in BS(Q)$ and $(j, k) \in IC(Q')$. We have the following three inequalities:

$$d_j \geq \min(C_i^+(Q), \max(d_i, C_i^+(Q) - p_j)), \quad (3.8)$$

$$p_j \leq p_k, \quad (3.9)$$

$$d_j \leq \max(d_k, C_k^-(Q')). \quad (3.10)$$

Obviously (3.8) implies $d_j \geq C_i^+(Q) - p_j$, and Lemma 5 guarantees $C_i^+(Q) - p_j \geq C_k^-(Q') + p_i$, so it follows that $d_j > C_k^-(Q')$. This inequality and (3.10) imply $d_j \leq d_k$. Using $d_k \geq d_j$, (3.8), (3.9) and Lemma 6, we obtain that

$$d_k \geq \min(C_i^+(Q), \max(d_i, C_i^+(Q) - p_k)) = \text{med}(C_i^+(Q) - p_k, d_i, C_i^+(Q)), \quad (3.11)$$

i.e., $(i, k) \in BS(Q) \subseteq DC(Q)$.

Case (iii): $(i, j) \in IC(Q)$ and $(j, k) \in BS(Q')$. Because of $j \in \max(Q')$, let

$$C = C_j^+(Q') = \sum_{s \in N} p_s. \quad (3.12)$$

According to Definition 2 and Lemma 6, we have $p_i \leq p_j$ and the following inequalities:

$$d_i \leq \max(d_j, C_j^-(Q)), \quad (3.13)$$

$$d_k \geq \min(C, \max(d_j, C - p_k)) = \text{med}(C - p_k, d_j, C). \quad (3.14)$$

Due to $(k, j) \notin Q$ implied by $(j, k) \in Q$, it holds that $C_j^-(Q) \leq C - p_k$. Thus from (3.13) it follows that $d_i \leq \max(C - p_k, d_j)$. Due to this inequality, $C_i^+(Q) \leq C$ and (3.14), using Lemma 6 we obtain (3.11).

Case (iv): $(i, j) \in BS(Q)$ and $(j, k) \in BS(Q')$. Let C be the same as in (3.12). Now we have (3.8) and (3.14). Due to Lemma 6, (3.8) implies $d_j \geq \min(d_i, C_i^+(Q))$. Due to this inequality, $C_i^+(Q) \leq C$ and (3.14), using Lemma 6 we obtain (3.11). \square

Lemma 7. Assume that $Q^* = \{Q_0, Q_1, \dots, Q_m\}$ is a PAPO system on N from $Q_0 = Q_{00}(N)$. Assume that $i \in \max(Q_m)$, $(j, i) \notin Q_m$ and $(i, j) \in DC(Q_m)$. Also assume that $j \notin \max(Q_m)$ and that (j, k) is the earliest primitive arc from j w.r.t Q^* . Then $(k, i) \notin Q_m$ and $(i, k) \in DC(Q_m)$.

Proof. In accordance with Lemma 3, there is $t \in [0, m)$ such that

$$j \in \max(Q_t) \setminus \max(Q_{t+1}), \quad (j, k) \in DC(Q), \quad (j, k) \in Q_{t+1} \subseteq Q_m.$$

Applying Theorem 1 for $Q = Q_m$ and $Q' = Q_t \subset Q$, we obtain that $(i, k) \in DC(Q_m)$. Furthermore, $(k, i) \notin Q_m$ is implied by $(j, k) \in Q_m$ and $(j, i) \notin Q_m$. \square

4. On induced partial orders

The concept of induced PO is the same as the concept of induced subgraph when we represent a PO by a preference graph, see [1]. Given $Q \in PO(N)$ and $N' \subset N$, then $Q' = Q \cap N'^2 \in PO(N')$, and Q' is called the induced PO of Q on N' . In this section, the induced PO is discussed in relation to the dominance condition and with the proper augmentations. The following lemma can be proved easily from Definition 2.

Lemma 8. Let $Q \in PO(N)$, $N' \subset N$ and $Q' = Q \cap N'^2$. Then

$$IC(Q) \cap N'^2 \supseteq IC(Q'), \quad (4.1)$$

$$BS(Q) \cap N'^2 \subseteq BS(Q'). \quad (4.2)$$

Lemma 9. Let $Q \in PO(N)$, $N' \subset N$ and $Q' = Q \cap N'^2$. If $N \setminus N' \subseteq \max(Q)$, then

$$IC(Q) \cap N'^2 = IC(Q'), \quad (4.3)$$

$$DC(Q) \cap N'^2 \subseteq DC(Q'). \quad (4.4)$$

Proof. We claim that under the assumptions it holds that

$$j \in N', s \in N', (s, j) \in Q' \Leftrightarrow j \in N', s \in N, (s, j) \in Q. \quad (4.5)$$

The “only if” part of (4.5) (from the left side to the right side) is obvious, since $N' \subset N$ and $Q' \subset Q$. As for the “if” part, when the right side of (4.5) holds, then $(s, j) \in Q$ implies $s \notin \max(Q)$, and thus $s \in N'$ follows from $N \setminus N' \subseteq \max(Q)$. So $(s, j) \in Q \cap N'^2 = Q'$. Therefore (4.5) is proved.

As a consequence of (4.5), $C_j^-(Q') = C_j^-(Q)$ holds for any $j \in N'$, so (4.3) is proved. Finally, (4.4) follows from (4.3) and (4.2) in Lemma 8. \square

The following lemma can be proved easily from Definition 5.

Lemma 10. Given $Q \in PO(N)$ and $R = Q \oplus (i, j)$. Assume that $r \neq i, r \in \max(Q)$ and $N' = N \setminus \{r\}$. Let $Q' = Q \cap N'^2$ and $R' = R \cap N'^2$. If $r = j$, then $R' = Q'$. If $r \neq j$, then $R' = Q' \oplus (i, j)$.

Theorem 2. Assume that Q_m is a proper augmentation PO with degree m from $Q_{00}(N)$ and that $r \in \max(Q_m)$. Let Q'_m be the induced PO of Q_m on $N' = N \setminus \{r\}$. Then Q'_m is a proper augmentation PO with degree m' from $Q_{00}(N')$, where $0 \leq m' \leq m$.

Proof. Let $Q^* = \{Q_0, Q_1, \dots, Q_m\}$ be the proper augmentation PO system from $Q_0 = Q_{00}(N)$, and let $i_l j_l$ be the primitive arc of the proper augmentation from Q_l to Q_{l+1} , where $l = 0, 1, \dots, m-1$. $r \in \max(Q_m)$ implies $i_l \neq r$, so we have that

$$Q_{l+1} = Q_l \oplus i_l j_l, \quad i_l \neq r, \quad i_l j_l \in DC(Q_l), \quad l = 0, 1, \dots, m-1. \quad (4.6)$$

Using Lemma 10 for Q_l and Q_{l+1} in (4.6), we obtain that

$$Q'_{l+1} = \begin{cases} Q'_l & \text{if } j_l = r, \\ Q'_l \oplus i_l j_l & \text{if } j_l \neq r, \end{cases} \quad (4.7)$$

where $l = 0, 1, \dots, m-1$. Applying Lemma 9, we obtain from (4.6) that

$$i_l j_l \in DC(Q'_l), \quad \text{if } j_l \neq r, l \in [0, m). \quad (4.8)$$

Thus it follows from (4.7) and (4.8) that Q'_m is a proper augmentation PO from $Q'_0 = Q_{00}(N')$ with degree m' , where m' is obviously the cardinality of the set $\{l | j_l \neq r, 0 \leq l < m\}$. \square

5. The main theorem and its proof

At first, we give the following three simple lemmas without proof.

Lemma 11. *Given $Q \in PO(N)$, $|N| = n$ and $k \in N$, then $k \in \max(Q)$ iff there exists $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n)) \in LE(Q)$ such that $\sigma(n) = k$.*

Lemma 12. *Given $Q \in PO(N)$, $r \in \max(Q)$, $N' = N \setminus \{r\}$ and $Q' = Q \cap N'^2$. Assume that $\sigma = \sigma'r$, where σ' is a sequence of N' . Then $\sigma \in LE(Q)$ iff $\sigma' \in LE(Q')$.*

Lemma 13. *Suppose that $TTP(N)$ has an optimal solution with job r in the last position. Let $N' = N \setminus \{r\}$. Then $\sigma = \sigma'r$ solves $TTP(N)$ iff σ' solves $TTP(N')$.*

Lemma 14. *Suppose that $Q^{**} = \{Q_0, Q_1, \dots, Q_m, Q_{m+1}\}$ is a $PAPQ$ system from $Q_0 = Q_{00}(N)$, and suppose that Q_m is a consistent PO of $TTP(N)$, then there exists an optimal solution σ for $TTP(N)$ such that $\sigma(n) \in \max(Q_{m+1})$.*

Proof. Let (i, j) be the primitive arc of the proper augmentation from Q_m to $Q_{m+1} = Q_m \oplus (i, j)$. So we have that $(i, j) \notin Q_m$ and

$$(j, i) \notin Q_m, \quad (i, j) \in DC(Q_m). \quad (5.1)$$

Let $\pi \in LE(Q_m)$ be an optimal solution for $TTP(N)$. We consider two cases as follows:

Case (i): $\pi(n) \neq i$. Let $r = \pi(n)$, it follows from $\pi \in LE(Q_m)$ and Lemma 11 that $r = \pi(n) \in \max(Q_m)$. Noticing $r = \pi(n) \neq i$ and using Lemma 2, we obtain that $r = \pi(n) \in \max(Q_{m+1})$. So in this case, just let $\sigma = \pi$, and the result is obtained.

Case (ii): $\pi(n) = i$. Similarly it follows from $\pi \in LE(Q_m)$ and Lemma 11 that

$$i = \pi(n) \in \max(Q_m). \quad (5.2)$$

Let r be $\text{terminal}(j)$ w.r.t $Q^* = \{Q_0, Q_1, \dots, Q_m\}$ according to Definition 9. And when $j \notin \max(Q_m)$, let (j, k) be the first arc in the primitive path from j to r , i.e., the earliest primitive arc from j w.r.t Q^* .

Because of (5.1) and (5.2), Lemma 7 applies and it holds for k that

$$(k, i) \notin Q_m, \quad (i, k) \in DC(Q_m). \quad (5.3)$$

Since (5.3) is in the same form as (5.1) with j replaced by k , Lemma 7 can be applied again if $k \notin \max(Q_m)$. Applying Lemma 7 successively along the primitive path from j to r , we eventually obtain that

$$(r, i) \notin Q_m, \quad (i, r) \in DC(Q_m). \quad (5.4)$$

Obviously (5.4) implies $r \neq i$, so according to Lemma 2 it follows from $r \in \max(Q_m)$ and $Q_{m+1} = Q_m \oplus (i, j)$ that

$$r \in \max(Q_{m+1}). \quad (5.5)$$

Due to (5.4) and $\pi^{-1}(r) < n = \pi^{-1}(i)$, let σ be the sequence obtained from taking the Emmons' transformation on π for job pair (i, r) according to Definition 3. Then $\sigma(n) = r$ satisfies (5.5), and σ is also an optimal solution for $TTP(N)$, since $T(\sigma) \leq T(\pi)$ holds according to the Transformation Theorem, and since the optimality of π is assumed. \square

Theorem 3. Suppose that Q is a proper augmentation PO on N from null. Then Q must be a consistent PO of $TTP(N)$.

Proof. We consider the theorem as a proposition involving two integer parameters, denoted by $CPO(|N|, M)$, where $|N|$ is the size of the job set N , and M stands for degree of the proper augmentations from null to Q . We prove that $CPO(|N|, M)$ is true in the domain D as follows:

$$D = \{(|N|, M) \mid |N| \geq 2, 0 \leq M \leq |N|(|N| - 1)/2\}, \quad (5.6)$$

where M is restricted with an upper bound, since at most Q includes $|N|(|N| - 1)/2$ ordered pairs composed of different elements of N , and since at least one ordered pair is added by a proper augmentation.

By "initial cases" we mean that $(|N|, M)$ is in D_0 as follows:

$$D_0 = \{(|N|, M) \mid |N| \geq 2, M = 0\} \cup \{(|N|, M) \mid |N| = 2, M = 1\}. \quad (5.7)$$

$CPO(|N|, M)$ is certainly true for initial cases $(|N|, M) \in D_0$, because $Q_{00}(N)$ for case $(|N|, M) = (|N|, 0)$ is a trivially consistent PO, and because Emmons' transformation must result in an optimal solution for case $(|N|, M) = (2, 1)$.

We are going to prove $CPO(|N|, M)$ by induction on $|N|$ and M simultaneously. As an induction hypothesis, $CPO(|N|, M)$ is assumed to be true for the following

two cases:

$$|N| = n - 1, \quad 0 \leq M \leq (n - 1)(n - 2)/2, \quad (5.8)$$

$$|N| = n, \quad M = m(0 \leq m < n(n - 1)/2), \quad (5.9)$$

Now we claim that under the induction hypothesis, $CPO(|N|, M)$ is also true for the case of $|N| = n$ and $M = m + 1$.

Let $Q^{**} = \{Q_0, Q_1, \dots, Q_m, Q_{m+1}\}$ be the proper augmentation PO system on N from $Q_0 = Q_{00}(N)$ with $|N| = n$, and let $Q = Q_{m+1}$. According to the induction hypothesis for case (5.9), Q_m is a consistent PO of $TTP(N)$, so Lemma 14 applies and there is an optimal solution for $TTP(N)$ with some r in the last position satisfying

$$r \in \max(Q). \quad (5.10)$$

Let $N' = N \setminus \{r\}$ and Q' be the induced PO of $Q = Q_{m+1}$ on N' . Theorem 2 ensures that Q' is a proper augmentation PO on N' with degree m' from $Q_{00}(N')$, where $0 \leq m' \leq (n - 1)(n - 2)/2$. Due to the induction hypothesis for case (5.8), Q' is a consistent PO of $TTP(N')$. So there exists an optimal solution σ' for $TTP(N')$ such that

$$\sigma' \in LE(Q'). \quad (5.11)$$

Noticing (5.10), (5.11) and the optimality of σ' , and applying Lemmas 12 and 13, we obtain that $\sigma = \sigma'r \in LE(Q) = LE(Q_{m+1})$ and that $\sigma = \sigma'r$ is an optimal solution for $TTP(N)$. Hence $Q = Q_{m+1}$ is a consistent PO of $TTP(N)$. Thus the proposition $CPO(n, m + 1)$ is proved.

Certainly, beginning from the “initial cases”, i.e., the cases for $(|N|, M)$ in D_0 defined by (5.7), the induction process (from cases (5.8) and (5.9) to the case of $|N| = n$ and $M = m + 1$) is able to run over the domain D defined by (5.6). Otherwise, let us consider a “minimal” case according to the lexicographical order of $(|N|, M)$ ($|N|$ first, M second) among all “negative” cases (i.e., $(|N|, M)$ for which $CPO(|N|, M)$ does not hold), and apply the induction process for the “minimal” case, then we would obtain a contradictory result. Hence $CPO(|N|, M)$ is true for any $(|N|, M) \in D$. \square

6. Concluding remarks

First, we mention that there exist examples which show that transitivity does not hold in general for the dominance condition $DC(Q)$, although the restricted transitivity for $DC(Q)$ is proved in Section 3.

Secondly, using a similar proof, the main result (Theorem 3) can be generalized to the following result: Suppose that Q is a proper augmentation PO on N from Q_0 , and that any primitive arc (i, j) of the proper augmentations satisfies either $j \in \max(Q_0)$ or $i \in \min(Q_0)$, then Q must be a consistent PO of $TTP(N)$. Furthermore, this result can

be generalized to and used for the total tardiness problem with given precedence constraints.

Thirdly, Example 1 in Section 2 indicates that a proper augmentation PO of a consistent PO is not necessarily a consistent PO , so in general the set $\mathcal{C}(N)$ composed of all consistent partial orders of $TTP(N)$ is not closed w.r.t. proper augmentations. On the other hand, let $\mathcal{A}(N)$ stand for the set of all proper augmentation partial orders from null, and then $\mathcal{A}(N)$ is obviously closed w.r.t. proper augmentations. Thus our main result can be restated as $\mathcal{A}(N) \subseteq \mathcal{C}(N)$. In relation to the above discussions, it might be possible to define a subset $\mathcal{B}(N)$ of $\mathcal{C}(N)$ by specifying some additional properties so that $\mathcal{B}(N)$ is closed w.r.t. proper augmentations and $\mathcal{B}(N)$ contains null. This remark concerns another idea for proving the main result of this paper.

Finally, the concept of consistent conditions can be defined for any optimization problem similarly as for $TTP(N)$, and obviously consistent conditions can serve as necessary conditions since usually it suffices to find one optimal solution only. For example, if some consistent conditions for an optimization problem have been verified, then in any branching procedure for the problem, all the branches disobeying the consistent conditions can be cut away with at least one optimal solution in the remaining branches. Thus the concept of consistent conditions has been actually used in many research papers. We think that the notion of “consistent conditions” can be used more consciously in optimization, and that the augmentation process discussed in this paper might be extended to certain consistent conditions for some different types of optimization problems.

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Appendix: Proofs of the Interchange Lemma and the Backward-Shift Lemma

The Interchange Lemma and the Backward-Shift Lemma in Section 1 are the underlying results which motivated the discussion about consistent partial orders and their proper augmentations for the total tardiness problem. These two important lemmas were obtained in Emmons [2], but were not stated independently. Actually the conclusions of them were contained implicitly in the proofs of some theorems in [2]. This is one of the reasons for us to give this appendix. The other reason is that the proofs here are simplified.

Lemma A1. *If $C \geq C'$, then it holds that*

$$\max(0, C - d) - \max(0, C' - d) = \max(0, \min(C - d, C - C')).$$

Proof. It can be checked easily for cases: (i) $d < C'$, (ii) $C' \leq d \leq C$, (iii) $d > C$. \square

Proof of the Interchange Lemma in Section 1. It is assumed that $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, $\sigma^{-1}(j) < \sigma^{-1}(i)$ and

$$p_i \leq p_j, \quad d_i \leq \max(d_j, C_j(\sigma)). \quad (\text{A.1})$$

Let p be the total processing time of all the jobs between j and i in job sequence σ . In relation to α obtained from σ by interchanging i and j , let C be the value given by

$$C = C_i(\sigma) = C_i(\alpha) + p + p_j = C_j(\sigma) + p + p_i = C_j(\alpha).$$

So $C - C_j(\sigma) = p + p_i$. Obviously (A.1) implies that

$$p + p_j \geq p + p_i, \quad C - d_i \geq \min(C - d_j, p + p_i). \quad (\text{A.2})$$

Using Lemma A1 we have that

$$T_i(\sigma) - T_i(\alpha) = \max(0, \min(C - d_i, p + p_j)), \quad (\text{A.3})$$

$$T_j(\alpha) - T_j(\sigma) = \max(0, \min(C - d_j, p + p_i)). \quad (\text{A.4})$$

From (A.2)–(A.4) it follows that $T_i(\sigma) - T_i(\alpha) \geq T_j(\alpha) - T_j(\sigma)$, and so $T_i(\alpha) + T_j(\alpha) \leq T_i(\sigma) + T_j(\sigma)$. Also, for $k \neq i, j$, $T_k(\alpha) \leq T_k(\sigma)$ holds because of $p_i \leq p_j$. Making summations of $T_s(\alpha)$ and of $T_s(\sigma)$ for $s \in N$, we obtain that $T(\alpha) \leq T(\sigma)$. \square

Proof of the Backward-Shift Lemma in Section 1. It is assumed that $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$, $\sigma^{-1}(j) < \sigma^{-1}(i)$ and that

$$d_j \geq \min(C_i(\sigma), \max(d_i, C_i(\sigma) - p_j)). \quad (\text{A.5})$$

Let p be the total processing time of all the jobs between j and i in job sequence σ . In relation to β obtained from σ by shifting j to be immediately after i , let C be given by

$$C = C_i(\sigma) = C_i(\beta) + p_j = C_j(\sigma) + p + p_i = C_j(\beta).$$

Obviously (A.5) implies that

$$C - d_j \leq \max(0, \min(C - d_i, p_j)). \quad (\text{A.6})$$

Using Lemma A1 we obtain that

$$T_i(\sigma) - T_i(\beta) = \max(0, \min(C - d_i, p_j)), \quad (\text{A.7})$$

$$T_j(\beta) - T_j(\sigma) = \max(0, \min(C - d_j, p + p_i)). \quad (\text{A.8})$$

From (A.6)–(A.8) it follows that $T_j(\beta) - T_j(\sigma) \leq \max(0, C - d_j) \leq T_i(\sigma) - T_i(\beta)$. The other parts of proving $T(\beta) \leq T(\sigma)$ is similar as in the proof of the Interchange Lemma. \square

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